

Averaging Einstein's Equations: The Linearized Case

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We introduce a simple and straight-forward averaging procedure, which is a generalization of one which is commonly used in electrodynamics, and show that it possesses all the characteristics we require for linearized averaging in general relativity and cosmology – for weak-field and perturbed FLRW situations. In particular we demonstrate that it yields quantities which are approximately tensorial in these situations, and that its application to an exact FLRW metric yields another FLRW metric, to first-order in integrals over the local coordinates. Finally, we indicate some important limits of any linearized averaging procedure with respect to cosmological perturbations which are the result of averages over large amplitude small and intermediate scale inhomogeneities, and show our averaging procedure can be approximately implemented by that of Zotov and Stoeger in these cases.

I. INTRODUCTION

It is usual in cosmology to consider the standard Universe as spatially homogeneous and isotropic on the largest scales. In fact, there is very good observational support for doing so. However, we also know that inhomogeneities exist at almost all scales – the smaller the scale the larger the inhomogeneity. All these inhomogeneities are apparently insignificant in cosmology, as long as we are not interested in modelling structure formation. It is therefore usual and considered acceptable by the vast majority of researchers to ignore them in investigating the dynamics and geometry of the Universe as a whole. Nevertheless, Ellis [1] has given compelling reasons why conceptually the large-scale cosmological metric should really be an average over very large regions of space-time. According to this view, the metric $g_{\mu\nu}$ and the Einstein field equations

$$G_{\mu\nu} \equiv R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = 8\pi T_{\mu\nu}, \quad (1)$$

where $R_{\mu\nu}$ is the Ricci tensor and R is the Ricci scalar, both depending on second-order derivatives of $g_{\mu\nu}$, and $T_{\mu\nu}$ is the stress-energy tensor, must be averaged over small and intermediate local inhomogeneities on larger and larger scales to obtain the average cosmological metric and the averaged dynamical equations.

However, averaging and operating with the Einstein differential operator on a metric do not commute, because of the nonlinearity of the operator. Thus, the solution to the averaged Einstein equations will *not* be the averaged metric. The averaged metric will, therefore, obey equations different from the averaged Einstein equations. In general, the smoothing-out operation will introduce extra tensor terms in the field equations, which may affect the dynamics and the energy conditions in the averaged universe. Furthermore and just as importantly, it is a very complicated and unresolved issue to define an adequate averaging scheme with the necessary properties – including the uniqueness of the averaged objects and their at least approximate tensorial character. This difficulty arises, because, in general, integrating a tensor field does not yield another tensor field in a curved space-time.

In spite of these difficulties and uncertainties, several averaging procedures have been proposed by Isaacson [2], Noonan [3,4], Zotov and Stoeger [5], and more recently by Boersma [6]. Their different philosophies and results, together with a comparison with the averaging scheme we shall introduce here, are briefly described and discussed below. Although far from the object of the present paper, it is also worth mentioning the ideas developed by Zalaletdinov in constructing his theory of macroscopic gravity [7]. He has used the concept of duality – the existence of two types of observers, microscopic and macroscopic – for the study of classical physical phenomena. Relying on this, he constructed a theory based on averaging a curved space-time itself and then determined the geometrical objects (metric, connection and curvature) which describe the averaged space-time [7]. This approach is non-perturbative in nature.

In this investigation we define an averaging scheme, which is a straight-forward generalization of one often used in macroscopic electromagnetic theory [8], carefully examine its properties, particularly for weak fields and for perturbations, and show that it is also an improvement of Noonan's averaging procedure [3,4]. The behavior of our

averaging scheme in the cases in which the linear approximation is sufficient is acceptable. In these situations the noncommutability of averaging and operating on the metric with the relevant differential operator is replaced by commutability, yielding simple, almost trivial results. However, these provide an essential reference for evaluating our averaging procedure and for reaching firm conclusions concerning the proper interpretation of and constraints on cosmological averaging in general, as well as interesting and important applications. Since much of cosmology involves perturbation treatments, we believe it is crucial to be clear about the averaging operator and its results in this context first, in order to be able to understand the cases in which nonlinear effects become important.

Next, we discuss in detail the general properties of our averaging operator – in particular its approximately tensorial character in cosmological coordinates and for general coordinates in the weak field case, and its apparent lack of uniqueness with respect to local coordinates.

Penultimately, we apply our averaging procedure to the important case of perturbed Friedmann-Lemaître-Robertson-Walker (FLRW) cosmologies, showing that averaging an exact FLRW space-time yields, to first order, in the local coordinates, an FLRW space-time, and further that averaging a perturbed FLRW cosmology yields, as expected, another perturbed FLRW cosmology. We then discuss in some detail, from the averaging point of view, what type of inhomogeneities in an FLRW background may be considered as perturbations of an FLRW metric. Our linear formalism is able to handle deviations for which the local dynamics are not completely decoupled from the general expansion of the universe. In contrast, if the local inhomogeneities are not in the linear regime, then a first order perturbative approach cannot be applied. Nevertheless, our general averaging scheme is still valid, but the resulting metric is not a simple superposition of FLRW plus perturbations.

Finally, we show how our averaging procedure may be approximately implemented in this nonperturbative case in simple situations. The procedure used by Zotov and Stoeger [5], applied throughout a space at each point, essentially accomplishes in an easily implementable way what our averaging procedure requires.

II. THE AVERAGING PROCEDURE

A. Definition

We define an averaging operator, or simply, an *averager*, acting on a field Q as

$$\langle Q(x) \rangle = \frac{\int Q(x+x') \sqrt{-g(x+x')} d\Omega'}{\int \sqrt{-g(x+x')} d\Omega'}. \quad (2)$$

This is a simple extension of the definition used by Jackson [8], for macroscopic electromagnetic fields. Here Q is any field – it may be a scalar, a vector, or a tensor. The coordinate x gives the cosmic, large-scale location of the averaging volume in space-time, and x' gives the small-scale location of a point within the averaging volume Ω' relative to its location x (its center, if it is a sphere) in the space-time, both expressed in Minkowski (rectangular) coordinates. Throughout this paper we often write x and x' for the cosmological and local coordinates, respectively, and $x+x'$ for their sum, by which we mean x^μ , x'^μ and $x^\mu + x'^\mu$, respectively. The metric of the space-time, for the observer who sees the inhomogeneities, is $g_{\mu\nu}(x+x')$ at the point $x+x'$. This is why the determinant of the metric g in the integrands of Eq. (2) depends on $x+x'$. Though each integration is only over the local regions of space-time dominated by the inhomogeneities, there are innumerable such averaging volumes spread over the observable universe – one for each cosmological coordinate x – and the metric within each one depends, of course, not just on the local coordinate x' but also on the cosmic location of the the averaging volume, given by the cosmological coordinate x . Obviously, the result of the averaging will depend strongly on the length scale over which it is performed. In general, that will be larger than, or of the order of, the characteristic length scale of the inhomogeneities over which we want to average.

This definition is not invariant under a change of coordinates. It can only be implemented as such in a coordinate system like Minkowski's in which the cosmological coordinates and the local coordinates are “parallel”, or translationally related. Nevertheless, as we shall see, our averager has some very nice properties. In particular, it is almost

invariant under transformations of the cosmological coordinates, and under transformations of the complete coordinate $x + x'$ in the weak field and perturbation cases.

We would like to stress that the quantities that are averaged are always referred to the same scale. In the process of averaging, the only observer involved is the one at the local ‘small scale’, which means that we are treating the problem in a self-consistent way. In other considerations one might have another observer – a cosmic observer – who does not see the inhomogeneities. In those cases, we would need to compare the results of the latter with the averaged ones.

There are two motivations for choosing this procedure for averaging. The first is simply that it implements what we intuitively envision as averaging over a given length scale, providing an assignment of an average value of a quantity to every point x throughout the space-time manifold taking into consideration the possibly different values of the metric in different cosmological locations, and specifying unambiguously the relationship between the large-scale cosmological coordinate system and the local, small-scale coordinate system, over which the averages are carried out. One simply takes the averaging volume and shifts it from point to point throughout the universe, averaging at each point according to Eq. (2). This is precisely what is done in making the transition from microscopic to macroscopic electromagnetism [8], as we have indicated above. Secondly, as we can explicitly show (see Section IV), the averaging procedure defined in Eq. (2) yields quantities which are approximately tensorial in character with respect to the cosmological coordinates x , and with respect to the full coordinates $x + x'$ for weak gravitational fields. That is, if $Q(x + x')$ is a tensor with respect to $x + x'$, $\langle Q(x) \rangle$ is almost a tensor of the same type with respect to x , and with respect to $x + x'$ for weak fields and perturbed space-times – deviating from a true tensor by only very small quantities. This is a very attractive property of our averager.

At the same time, there are some other aspects of our averager which, at first sight, are not so attractive and require further explanation. First, as we have already mentioned, the relationship between the cosmological and the local coordinate systems, as formally expressed in Eq. (2), can only be realized in a very restricted class of coordinate systems – those in which they can be related to one another by simple translation, so that the “complete” coordinate of any point can be expressed simply as the sum of the cosmological coordinates and the local coordinates at that point. Most coordinate systems will not fulfil this requirement. This may not be a problem, since, within the demands of coordinate covariance, we are allowed to choose any coordinate system we want. In particular, any coordinate system which simplifies the formulation of our problem may be selected – as long as the quantities in question are tensors, or nearly tensors, thus assuring us that they are equivalent to their forms in more complicated coordinate systems. For other choices we could also specify the relation between the cosmological coordinates and the local coordinates, but the relationship between the two would, in general, be much more complicated – not specifiable by a simple sum of the two coordinates, that is by simple translation.

This relation between the cosmological coordinates and the local coordinates leads another potential difficulty. Although it is clear that the average we have defined in Eq. (2) is approximately covariant with respect to the cosmological coordinates [9], as we shall show in Section V, strictly speaking a transformation of the cosmological coordinates x in the integrals of Eq. (2) should be accompanied by a transformation of the x' coordinates, along with an induced change in their functional relationship (from a simple sum to something more complicated), as indicated above. However, even then the averaging over the local coordinates would still not be invariant under those changes. In light of the arguments given by Ellis and Matravers [10] and Ellis, Matravers and Zalaletdinov [11] for using preferred coordinate systems in general relativity which are appropriate to the physical situation and simplify the problem, even though they break coordinate invariance, we maintain that this lack of covariance of the averaging procedure with respect to the local coordinates should not be considered an essential problem in most cosmological applications. As we shall show, this is certainly the case for weak gravitational fields and for perturbations from an exact cosmological solution (e.g. FLRW) to the field equations. In these cases the deviations caused by the lack of coordinate covariance of the integrals with respect to the local coordinates are small. Even for more general cases, it is likely that these deviations will be small, as long as we are averaging over volumes for which the space is almost flat – for which the length scale of the averaging volumes is smaller than the radius of curvature of the universe. Thus, in that sense the most important property of our averaging procedure is that it is covariant relative to transformations of the cosmological coordinates.

Another way of expressing the above objection is that in carrying out our averaging procedure we are effectively adding vectors and tensors at different points, which is not a well-defined, or even an allowed, operation on vectors

and tensors if the space-time is curved. We should really incorporate bi-vectors in the averaging integrals in order to translate vectors and tensors to the point designated by the cosmological metric and then add them, as proposed by Isaacson [2]. The principal reason why we have not done this is that there is no easy way of defining the needed bi-vectors without already knowing the cosmological background metric, and it is precisely this cosmological background metric that we eventually want to determine by carrying out the averaging over small scales. We discuss this in more detail in the next subsection below. (Even though throughout most of this paper we limit ourselves to cases in which we have a well-defined background – the weak-field and perturbed FLRW situations – we want to be able to use our averager in more general cases. We shall present one way of doing that approximately in the last section.) Furthermore, as we have just discussed, in the case of weak fields and of almost flat averaging volumes, the errors introduced by neglecting the bivectors will be small. Zalaletdinov [7] constructs his macroscopic theory of gravity with bi-vectors, but these have to be solved consistently with the other field equations. This is a possible way of proceeding, but leads to such a complicated theory that solutions in the simplest cases have yet to be obtained.

B. Comparison with other definitions

There are a number of other definitions of averaging which have been suggested. We describe them briefly here, and compare them with ours, indicating its advantages.

- Noonan's Operator

The averaging operator given by Noonan [3,4] is

$$\langle Q' \rangle = \frac{\int Q'(x') \sqrt{-g(x')} d\Omega'}{\int \sqrt{-g} d\Omega'}, \quad (3)$$

where

- $\sqrt{-g}$ refers to the determinant of a macroscopic metric.
- The region of integration is finite and its size is bigger than the usual scale of the microscopic observer and smaller than that of the macroscopic observer. Using this fact the denominator may be approximated by $\int d\Omega$ in the cases in which the variation of g is not abrupt.
- The dependence of $\langle Q' \rangle$ on the space-time coordinates comes from using indefinite integrals. This means that the variable in the averager is now the boundary of the integration volume – which, we believe, is not well-defined for an averager.

The three points mentioned above represent the differences from our definition of the averaging procedure. One possible weakness of Noonan's approach (for details, see also [12]) is the simultaneous introduction of two scales in his averages: the metric in his definition is one of large scale, whereas his averaging quantities are over small scales.

- Isaacson's Operator

Isaacson's procedure of averaging [2] directly deals with the problem that, in general, the result of integrating a tensor field does not give another tensor, because tensors at different points have transformation properties which depend on their location, as we have already indicated. Since one can only add tensors at the same point, the objective is to carry them to a certain common point and to add them there. To do that, one has to introduce the *bi-vector of parallel displacement* $j^\beta_\alpha(x, x')$ [13–15]. This object transforms as a vector with respect to coordinate transformations at x or at x' , and it has the property that, given a vector (or tensor, in general) A_β at x' , then $A_\alpha(x) = j^\beta_\alpha(x, x') A_\beta(x')$ is the unique vector at x that can be obtained by parallel displacement

of A_β from x to x' along a geodesic.¹

Given a tensor $T_{\mu\nu}$ Isaacson's averaging operator is defined as:

$$\langle T_{\mu\nu}(x) \rangle = \int_{all\ space} j_\mu^{\alpha'}(x, x') j_\nu^{\beta'}(x, x') T_{\alpha'\beta'}(x') f(x, x') d^4 x' \quad (4)$$

where $f(x, x')$ is a weighting function which falls smoothly to zero when x and x' differ by a distance greater than d , its integral is normalized to one over all space-time. This definition carries with it all the properties of the bi-vector of parallel geodesic displacement, for example

$$j_\mu^{\alpha'}(x, x') j_\nu^{\beta'}(x, x') g_{\alpha'\beta'}(x') = g_{\mu\nu}(x). \quad (5)$$

This implies that if we select $g_{\alpha'\beta'}(x')$ to be the small-scale metric, it is impossible to obtain an averaged metric (corresponding to cosmological scales) different from the assumed large-scale metric. That is,

$$\begin{aligned} \langle g_{\mu\nu}(x) \rangle &= \int_{all\ space} j_\mu^{\alpha'}(x, x') j_\nu^{\beta'}(x, x') g_{\alpha'\beta'}(x') f(x, x') d^4 x' = \\ &= g_{\mu\nu}(x) \int_{all\ space} f(x, x') d^4 x' = g_{\mu\nu}(x). \end{aligned} \quad (6)$$

This represents a crucial problem for averaging, since, as we have stressed, we ideally want to obtain metrics on larger scales as averages of the small-scale metrics, without assuming the particular form of the large-scale metrics.

- **Zotov and Stoeger Averaging**

Zotov and Stoeger first presented a simple procedure in which they average over an unbound distribution of stars –in a static background– and of galaxies –in an expanding background [5]. They use an averaging scheme which recalls a three dimensional version of finding a running mean with the simplest elementary volume: a sphere. Later, using similar techniques they construct averages over hierarchies of Swiss-cheese regions in elementary cosmological cells, which are the smallest volumes which are expanding with the Hubble flow. Though, they do not provide an averaging procedure which is specifically defined at each point of cosmological space-time, but rather construct averages over simple inhomogeneous configurations which they assign to entire regions, one can argue that over large volumes this procedure approximately gives an average at each point. In the examples treated, the metric (Swiss-cheese) of the inhomogeneities is considered spherically symmetric and therefore either Schwarzschild or FLRW. The attempt is to construct the physical large-scale background metric from averages over inhomogeneities which may be very large in amplitude compared to the provisional background. These averages are obtained by adding the averages over each inhomogeneity to one another along with the averages over the provisional “background” space in between the homogeneities. Bi-vectors are not used. We shall come back to discuss this scheme more fully at the end of the paper, as it can be construed as an approximate implementation of our averaging procedure in simple non-perturbative averaging situations.

- **Buchert and Ehlers averaging of Newtonian Cosmologies**

With a similar approach to that proposed by Noonan, Buchert and Ehlers [16,17] have focused on the averaging problem applied to the cosmological expansion of the Universe. They restrict themselves to the Newtonian approach, i.e. the analog of Friedmann's equation for the motion of self-gravitating pressureless fluid. They propose that the spatial average of a tensor field A in the domain $D(t)$ should be:

$$\langle A \rangle_D = \frac{1}{V} \int_D d^3 x A. \quad (7)$$

¹This is not true if there is more than one geodesic which joins the two given points x and x' , because the transport along each different geodesic varies. One may avoid such problems arguing that the two points are close enough to each other as to permit the existence of only one geodesic joining them.

Since the comoving volume $D(t) = a_D^3(t)$, the fluid elements move on the average according to

$$\langle \theta \rangle_D = \frac{\dot{V}}{V} = 3 \frac{\dot{a}_D}{a_D}. \quad (8)$$

With this in mind, $a_D(t)$ becomes the new scale factor and is shown to obey an *averaged Raychaudhuri equation*. Note that there are no spatial dependencies in the averaged quantities, only time is left as a variable. This means that the averaging scheme yields the same result at all positions in the synchronous gauge.

They also propose a scheme for general relativity which includes the determinant of the spatial metric $g = \det g_{ij}$ in the region of interest:

$$\langle A \rangle_D = \frac{1}{V} \int_D d^3x \sqrt{g} A \quad (9)$$

with $V = \int_D d^3x \sqrt{g}$, which resembles Noonan's definition even more closely. This average was then used by Russ et al. [18] to compute the effect of inhomogeneities on the age of a flat Universe. The sources of the inhomogeneities were taken from the Zel'dovich approximation to second order, so that their results are valid in an early stage of the evolution of the flat Universe. They conclude that the effect on the age of the universe is very small on those scales.

- **Boersma's Averaging Procedure**

Boersma [6] derived, from basic assumptions, a generic linearized spatial averaging operation for metric perturbations from FLRW, satisfying the condition that unperturbed FLRW is a stable fixed point of the averaging (that is, the averaging operation does not introduce spurious perturbations in the averaged metric). He specified the correspondence among points in the real spacetime, the averaged spacetime and the background spacetime, by the introduction of a bi-tensor density in the averaging integrals, which fulfils the same function as the bi-vectors in Isaacson's approach. He succeeded in deriving a general form of this bi-tensor density in terms the background metric and the future-directed unit vectors normal to the space-like surfaces over which the averages are being performed. With this formalism, and using Bardeen's [19] gauge-invariant quantities Boersma was able to resolve the gauge problem in his averaging procedure and apply it to the constraint equations on spacelike hypersurfaces, which are closely related to the generalized Friedmann equation.

III. THE WEAK FIELD LIMIT

Using our averager defined in Eq. (2) we begin to examine its properties in a very simple, almost trivial way, by considering its application to gravity in the weak field limit. We do this in order to assure ourselves that our averaging procedure fulfils the simplest intuitive requirements and to establish some results which we shall need later in applying it to perturbed FLRW cosmologies.

Let us consider a Minkowskian line element plus small corrections. That is,

$$g_{\mu\nu}(x) = \eta_{\mu\nu}(x) + h_{\mu\nu}(x), \quad |h_{\mu\nu}| \ll 1. \quad (10)$$

Applying our averager to this metric we have

$$\langle g_{\mu\nu}(x) \rangle = \frac{\int d\Omega' \sqrt{-g(x+x')} g_{\mu\nu}(x+x')}{\int d\Omega' \sqrt{-g(x+x')}}. \quad (11)$$

Since in the weak field limit (see below)

$$\sqrt{-g(x+x')} = 1 + \frac{1}{2} \eta^{\mu\nu} h_{\mu\nu}(x+x'), \quad (12)$$

and considering that the product of two or more elements of the matrix of perturbations is negligible we find, to first (linear) order,

$$\langle g_{\mu\nu}(x) \rangle = \eta_{\mu\nu} + \frac{\int d\Omega' h_{\mu\nu}(x+x')}{\int d\Omega'} + \mathcal{O}(h^2). \quad (13)$$

Thus, in general, averaging a Minkowski space + perturbations gives another perturbed Minkowski space – a result which is obvious, but reassuring. It is, of course, possible in some circumstances, that the second term of Eq. (9) will be zero, that is, the perturbations will average out over larger scales. For instance, “overdensities” may be partially or completely compensated by “underdensities”.

Furthermore, it is clear that in this weak field case – and in the case of perturbations to FLRW discussed in Section V – the operation of averaging *will* commute with constructing the Einstein field equations, since these are now linear. Thus, averaging will not introduce any new terms in the macroscopic field equations themselves, as it does in the exact case [1,5,12].

Let us now consider a region in space, free of gravitational sources but filled with gravitational radiation coming from a source situated at infinity. In these circumstances we can apply the weak field limit of Einstein’s equations

$$\nabla h_{\mu\nu} = 0, \quad (14)$$

$$\frac{\partial h_{\nu}^{\mu}}{\partial x^{\mu}} = \frac{1}{2} \frac{\partial h_{\mu}^{\mu}}{\partial x^{\nu}}. \quad (15)$$

Let us now suppose that we have two scales, a macroscopic and a microscopic scale, both free of sources. Then on both scales we will see the same behavior, i.e. on each scale we can write

$$h_{\mu\nu}^{(i)} = e_{\mu\nu}^{(i)} e^{ik_{\lambda}^{(i)} x^{(i)\lambda}} + C.C. \quad (16)$$

for $i = 1, 2$, where the $e_{\mu\nu}^{(i)}$ are the polarization tensors in each region. The microscopic scale is associated with our definition and the macroscopic one ought to be compared with the averaged result.

Recalling the definition of the average in the weak field limit,

$$\langle h_{\mu\nu}(x) \rangle = \frac{\int d^4x' \sqrt{-g(x+x')} h_{\mu\nu}(x+x')}{\int d^4x' \sqrt{-g(x+x')}}, \quad (17)$$

where $g = \det g_{\mu\nu} = \det(\eta_{\mu\nu} + h_{\mu\nu})$. Writing Eq. (10) in matrix notation allows us to compute g using

$$\ln \det(1 + A) = \text{tr} \ln(1 + A). \quad (18)$$

by properly defining A in terms of $h_{\mu\nu}$. Thus

$$\begin{aligned} \text{tr} \ln(1 + A) &= \sum_{\alpha} \ln(1 + A)_{\alpha\alpha} \\ &= \ln(1 + h_{00}) + \ln(-1 + h_{11}) + \ln(-1 + h_{22}) + \ln(-1 + h_{33}) \\ &= \ln(1 + h_{00})(-1 + h_{11})(-1 + h_{22})(-1 + h_{33}) \\ &\approx \ln(-1 - h_{00} + h_{11} + h_{22} + h_{33}), \end{aligned} \quad (19)$$

and

$$-g = 1 + h_{00} - h_{11} - h_{22} - h_{33}, \quad (20)$$

or

$$\sqrt{-g} \approx 1 + \frac{1}{2} \eta^{\mu\nu} h_{\mu\nu}. \quad (21)$$

This result is independent of the choice of the gauge, and the only hypothesis used is that $h_{\mu\nu}$ is small. Going back to the definition of the average, and using the property just demonstrated, we find to first order in h

$$\langle h_{\mu\nu}(x) \rangle = \frac{\int d^4x' h_{\mu\nu}(x+x')}{\int d^4x'}. \quad (22)$$

Therefore, using Eq. (16)

$$\langle h_{\mu\nu}(x) \rangle = \frac{1}{\Omega'} \left[e_{\mu\nu} \int d^4x' e^{ik_\lambda (x^\lambda + x'^\lambda)} + e_{\mu\nu}^* \int d^4x' e^{-ik_\lambda (x^\lambda + x'^\lambda)} \right], \quad (23)$$

or

$$\langle h_{\mu\nu} \rangle = c_1 e_{\mu\nu} e^{ik_\lambda x^\lambda} + C.C. \quad (24)$$

First of all, note that this averaged quantity transforms as a tensor in the weak field limit. Second the polarization tensors can now be redefined as they appear multiplied by a constant which contains information about the microscopic scales of the system. Thirdly, we have found that the averaged solution (i.e. the average of the solution in one scale) coincides with the solution on another scale. This was expected, because we have worked in the linear theory. There Einstein equations are linearized, and therefore, as we have already indicated, the averaging procedure does not introduce extra terms in them. This means, of course, that the averages are also solutions of the Einstein weak field equations.

We will now extend this analysis to see what happens in the weak field limit, when considering a region in which there is a source of gravitational radiation. In this case, one scale would correspond to that of the source, while the other could be thought of as the wave zone – that is, a scale in which the length scales are much larger than the size of the source. When averaging, we should compare the average metric $\langle h_{\mu\nu} \rangle$ with the one obtained for the wave zone.

The solution of the weak field Einstein equations in the presence of a source is the metric [20]

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (25)$$

with

$$h_{\mu\nu}(x) = 4G \int d^3x' \frac{S_{\mu\nu}(\vec{x}', t - |\vec{x} - \vec{x}'|)}{|\vec{x} - \vec{x}'|}. \quad (26)$$

Combining this Eq. with Eq. (17) we can write (to linear order in h)

$$\langle h_{\mu\nu}(x) \rangle = \frac{4G}{\Omega'} \int d^3x' d^4x'' \frac{S_{\mu\nu}(\vec{x}', t + t'' - |\vec{x} + \vec{x}'' - \vec{x}'|)}{|\vec{x} + \vec{x}'' - \vec{x}'|}. \quad (27)$$

If we now express the energy-momentum tensor as a Fourier integral, we can analyze each Fourier component separately, and integrate (or add those components) afterwards. This means that we can replace in Eq. (27), $S_{\mu\nu}(\vec{x}, t) = \hat{S}_{\mu\nu}(\vec{x}, \omega) e^{-i\omega t}$, i.e.

$$\langle h_{\mu\nu}(x) \rangle = \frac{4G}{\Omega'} \int d^3x' d^4x'' \hat{S}_{\mu\nu}(\vec{x}', \omega) \frac{e^{-i\omega(t+t''-|\vec{x}+\vec{x}''-\vec{x}'|)}}{|\vec{x} + \vec{x}'' - \vec{x}'|} + C.C. \quad (28)$$

Hereafter, we drop the complex conjugate of the quantities to make the expressions look simpler. However, they should be recalled at the end of the calculations, as all quantities are real. So far the only approximation made was to consider a weak field limit.

The scales in the problem are three:

1. Scale of the source, given by x' .
2. Scale of the averaging procedure (or *small scale*), given by x'' .
3. Scale of the metric (*large scale*), given mainly by x .

This means that scale (1) should be comparable with scale (2), and both much smaller than scale (3). Under this assumption we can perform a multipole expansion, which to first order is

$$|\vec{x} + \vec{x}'' - \vec{x}'| \approx |\vec{x} - \vec{x}'| + \vec{x}'' \cdot \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|}. \quad (29)$$

Thus, we can integrate over the *small scale* and obtain for the spatial part

$$I = \int d^3 x'' \frac{e^{i\omega|\vec{x} + \vec{x}'' - \vec{x}'|}}{|\vec{x} + \vec{x}'' - \vec{x}'|} = \frac{e^{i\omega|\vec{x} - \vec{x}'|}}{|\vec{x} - \vec{x}'|} \int d^3 x'' e^{i\omega \vec{x}'' \cdot \frac{(\vec{x} - \vec{x}')}{|\vec{x} - \vec{x}'|}}, \quad (30)$$

or

$$I = 4\pi \frac{e^{i\omega|\vec{x} - \vec{x}'|}}{\omega|\vec{x} - \vec{x}'|} \left[\frac{-1}{\omega} R \cos \omega R + \frac{1}{\omega^2} \sin \omega R \right], \quad (31)$$

where R is the size of the region over which we are averaging. As we stated before, R should be comparable to the size of the source, and much smaller than the overall scale of the problem. Substituting in Eq. (28), we obtain the averaged metric as

$$\begin{aligned} \langle h_{\mu\nu}(x) \rangle &= \frac{16\pi G}{\Omega'} \int dt'' e^{-i\omega(t+t'')} \times \\ &\int d^3 x' \frac{e^{i\omega|\vec{x} - \vec{x}'|}}{\omega|\vec{x} - \vec{x}'|} \left[\frac{-1}{\omega} R \cos \omega R + \frac{1}{\omega^2} \sin \omega R \right] \hat{S}_{\mu\nu}(\vec{x}', \omega). \end{aligned} \quad (32)$$

Applying again the multipole expansion for $|\vec{x}'| \ll |\vec{x}|$, finally

$$\langle h_{\mu\nu}(x) \rangle = \frac{16\pi G}{\Omega'} e^{-i\omega t} \frac{e^{i\omega r}}{r} \frac{1 - e^{-i\omega T}}{i\omega} \left[\frac{-1}{\omega} R \cos \omega R + \frac{1}{\omega^2} \sin \omega R \right] \int d^3 x' e^{-i\omega \vec{x}' \cdot \frac{\vec{x}}{r}} \hat{S}_{\mu\nu}(\vec{x}', \omega), \quad (33)$$

or,

$$\langle h_{\mu\nu}(x) \rangle = e^{ik_\lambda x^\lambda} e_{\mu\nu}(\vec{x}, \omega) + C.C. \quad (34)$$

with

$$e_{\mu\nu}(\vec{x}, \omega) = \frac{4G}{r} \frac{4\pi G}{\Omega'} \frac{1 - e^{-i\omega T}}{i\omega} \left[\frac{-1}{\omega} R \cos \omega R + \frac{1}{\omega^2} \sin \omega R \right] \int d^3 x' e^{-i\omega \vec{x}' \cdot \frac{\vec{x}}{r}} \hat{S}_{\mu\nu}(\vec{x}', \omega). \quad (35)$$

Here $e_{\mu\nu}$ is the polarization tensor that an observer in the wave zone would detect.

Note the contribution from the integration over the time t'' , $(1 - e^{-i\omega T})/i\omega$, with T being the size of the time interval in space-time. Certainly one expects the choice of the limit of integration T to be dependent on the nature of the problem. Let us suppose that only one frequency ω is being emitted by a source. Therefore, a suitable choice for T will be $\frac{\pi}{\omega}$. That is, T is basically the characteristic period of the system. Note as well that we have only taken into account characteristic scales of the system which are observed from the microscopic perspective.

Notice, too, that the averaged metric is, again to first order, a tensor. And even more, it has the same form as the solution of Einstein's equations in the weak field limit considered in the wave zone. This was to be expected, because we are still dealing with the linearized equations, and thus, the averaging is a linear procedure. That is, averaging the equations is still equivalent to averaging the metric, just because we are working in the weak-field limit.

IV. THE APPROXIMATE TENSORIAL CHARACTER OF THE AVERAGED QUANTITIES

In this section we examine whether or not the quantities defined by our averaging procedure are generally, or approximately, tensors under certain conditions.

We first analyze the tensorial status of our averaging procedure in the completely general case. In order to have $\langle L_{\mu\nu} \rangle$ as a tensor we would need the transformation of $L_{\mu\nu}$ such that

$$L'_{\mu\nu}(y+x') = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} L_{\alpha\beta}(x+x') \quad (36)$$

under a completely general function $x^\mu = f^\mu(y)$. If Eq. (36) is valid, we can take the derivatives outside the integral of our averaging definition and find that the averaged quantities behave as tensors too. To satisfy Eq. (36) we need to make a generic transformation between $z = x + x'$ and $z' = y + x'$, with a fixed x' . But defining $z' = h(z)$ in this way is equivalent to imposing very specific constraints on the original function f :

$$z' = y + x' = f(x) + x' = h(x + x').$$

This is not possible for every conceivable f , and thus the averager is not in general a tensor.

Even when we only transform the cosmological coordinate x , so that the averaging procedure yields almost tensorial quantities with respect to the cosmological coordinates only (the averaging integrations are only over x' , so that the transformations of the cosmological coordinates x can be taken through the integrals), [9] there is an important problem. Strictly speaking, we cannot transform the cosmological coordinates without also transforming the local coordinates and altering the functional relationship connecting them. Furthermore, the integral over the local coordinates x' is not covariant with respect to those changes.

Can we somehow show that the average is *approximately* coordinate covariant in some cases? That is indeed true for the weak-field and the perturbed-FLRW cases.

Let $L_{\mu\nu}$ be a tensor. In the weak field limit our averager applied to $L_{\mu\nu}$ takes the form

$$\langle L_{\mu\nu}(x) \rangle = \frac{\int d\Omega' (1 + \frac{1}{2} \eta^{\alpha\beta} h_{\alpha\beta}(x+x')) L_{\mu\nu}(x+x')}{\int d\Omega' (1 + \frac{1}{2} \eta^{\alpha\beta} h_{\alpha\beta}(x+x'))}. \quad (37)$$

To first order in the perturbation this becomes

$$\begin{aligned} \langle L_{\mu\nu}(x) \rangle &= \frac{1}{\Omega'} \int d\Omega' L_{\mu\nu}(x+x') + \frac{1}{\Omega'} \int d\Omega' \frac{1}{2} \eta^{\alpha\beta} h_{\alpha\beta}(x+x') L_{\mu\nu}(x+x') \\ &\quad - \frac{1}{\Omega'^2} \int d\Omega' L_{\mu\nu}(x+x') \int d\Omega'' \frac{1}{2} \eta^{\alpha\beta} h_{\alpha\beta}(x+x''). \end{aligned} \quad (38)$$

where $\Omega' = \int d\Omega'$, is the 4-volume and a real number which only depends on scale of the averaging.

We now employ Isaacson's averaging procedure and compare it with our own. In the Isaacson's case we have the averaging definition given in Eq. (4), which yields a *bona fide* tensor. We also know that the bi-vectors $j_\mu^{\alpha'}(x, x')$ must satisfy Eq. (5). In the weak field and perturbed space-time cases this implies that

$$j_\mu^{\alpha'}(x, x') = \delta_\mu^{\alpha'} - (1/2) h_\mu^{\alpha'}(x, x'). \quad (39)$$

Thus,

$$\langle T_{\mu\nu} \rangle_I = \int d^4x' T_{\mu\nu}(x') f(x, x') + O(1), \quad (40)$$

where $O(1)$ refers to integrals of first order in h . Thus, the first term in this equation is almost a tensor –differs from a tensor by small terms of order h . Then, our averaging procedure gives an integral like the first term of Eq. (40), except that it contains a $\sqrt{-g}$ instead of $f(x, x')$ –which makes no difference in the tensorial character of the integral in the weak field and perturbed field cases. Thus, our procedure also yields an approximate tensor in both of these cases, relative to the complete coordinate $x + x'$.

V. THE FLRW METRIC + PERTURBATIONS

Let us now consider an FLRW space-time with perturbations and apply the averager to the perturbed metric

$$g_{\mu\nu}(x) = g_{\mu\nu FLRW}(x) + h_{\mu\nu}(x), \quad (41)$$

where the FLRW line element in Cartesian coordinate can be written [21]

$$ds^2 = dt^2 - \frac{a^2(t) [dx^2 + dy^2 + dz^2]}{\{1 + \frac{1}{4}k[x^2 + y^2 + z^2]\}^2}. \quad (42)$$

Applying the averaging, we find

$$\langle g_{\mu\nu}(x) \rangle = \frac{\int g_{\mu\nu}(x+x') \sqrt{-g(x+x')} d\Omega'}{\int \sqrt{-g(x+x')} d\Omega'}. \quad (43)$$

Using Eq. (18) the determinant of the perturbed metric is, to first order in $h_{\mu\nu}$

$$\det g_{\mu\nu} = \det g_{\mu\nu FLRW} (1 + h_{\mu\nu} g_{FLRW}^{\mu\nu}), \quad (44)$$

and replacing this in Eq. (43) we obtain

$$\langle g_{\mu\nu}(x) \rangle = \frac{\int (g_{\mu\nu FLRW}(x+x') + h_{\mu\nu}(x+x')) \sqrt{-g_{FLRW}(x+x') (1 + h_{\mu\nu} g_{FLRW}^{\mu\nu})} d\Omega'}{\int \sqrt{-g_{FLRW}(x+x') (1 + h_{\mu\nu} g_{FLRW}^{\mu\nu})} d\Omega'}. \quad (45)$$

A. What is a perturbation?

Here we shall define more precisely what we mean by a perturbation to an FLRW metric. We will say that $h_{\mu\nu}$ in Eq. (41) is a perturbation if, just as in Eq. (10),

$$|h_{\mu\nu}| \ll 1. \quad (46)$$

Thus, if $h_{\mu\nu}$ is a perturbation to FLRW in one coordinate system, then in any other coordinate system, in which we can always write the metric as

$$g'_{\mu\nu}(y) = g'_{\mu\nu FLRW}(y) + h'_{\mu\nu}(y), \quad (47)$$

$h'_{\mu\nu}$ is also a perturbation. Though $h'_{\mu\nu}$ will naturally be different from $h_{\mu\nu}$, it will always satisfy Eq. (46), if $h_{\mu\nu}$ itself does so. That is simply because, if we apply a general coordinate transformation

$$g'^{\alpha\beta}(y) = \frac{\partial y^\alpha}{\partial x^\mu} \frac{\partial y^\beta}{\partial x^\nu} g^{\mu\nu}(x) \quad (48)$$

to Eq. (41), we trivially find that the coordinate transformation Eq. (48) also relates the $h'_{\mu\nu}$ to the $h_{\mu\nu}$, and that, therefore, the $h'_{\mu\nu}$ will be small, if the $h_{\mu\nu}$ are. Thus also the determinate of the transformed metric given in Eq. (47) will be given by Eq. (44), with the unprimed metric variables just replaced by the primed (transformed) metric variables.

B. Averaging FLRW Perturbations

Let us now obtain a general expression for the average of the perturbed metric. From Eq. (45)

$$\langle g_{\mu\nu} \rangle = \langle g_{\mu\nu FLRW} + h_{\mu\nu} \rangle = \frac{N}{V}, \quad (49)$$

with

$$\begin{aligned}
N = & \int d^4x' \sqrt{-g_{FLRW}(x+x')} g_{\mu\nu FLRW}(x+x') + \int d^4x' \sqrt{-g_{FLRW}(x+x')} h_{\mu\nu}(x+x') \\
& + \int d^4x' \sqrt{-g_{FLRW}(x+x')} \frac{1}{2} g_{FLRW}^{\alpha\beta}(x+x') h_{\alpha\beta}(x+x') g_{\mu\nu FLRW}(x+x'),
\end{aligned} \tag{50}$$

and

$$V = \int d^4x' \sqrt{-g_{FLRW}(x+x')} \left(1 + \frac{1}{2} g_{FLRW}^{\alpha\beta}(x+x') h_{\alpha\beta}(x+x')\right). \tag{51}$$

The denominator can be written as a product, the first term being the invariant volume for an FLRW space-time. Expanding the rest of the terms to first order in $h_{\mu\nu}$ we get

$$\begin{aligned}
\langle g_{\mu\nu} \rangle = & \frac{\int d^4x' \sqrt{-g_{FLRW}(x+x')} g_{\mu\nu FLRW}(x+x')}{\int d^4x' \sqrt{-g_{FLRW}(x+x')}} + \frac{\int d^4x' \sqrt{-g_{FLRW}(x+x')} h_{\mu\nu}(x+x')}{\int d^4x' \sqrt{-g_{FLRW}(x+x')}} \\
& + \frac{\int d^4x' \sqrt{-g_{FLRW}(x+x')} \frac{1}{2} g_{FLRW}^{\alpha\beta}(x+x') h_{\alpha\beta}(x+x') g_{\mu\nu FLRW}(x+x')}{\int d^4x' \sqrt{-g_{FLRW}(x+x')}} \\
& - \int d^4x' \sqrt{-g_{FLRW}(x+x')} g_{\mu\nu FLRW}(x+x') \times \\
& \times \frac{\int d^4x' \sqrt{-g_{FLRW}(x+x')} \frac{1}{2} g_{FLRW}^{\alpha\beta}(x+x') h_{\alpha\beta}(x+x')}{\left(\int d^4x' \sqrt{-g_{FLRW}(x+x')}\right)^2}.
\end{aligned} \tag{52}$$

We are now in a position to show that, for our averaging procedure, the average of an FLRW metric itself is, to first order in the local coordinates, always an FLRW metric, and further that the average metric of a perturbed FLRW metric is also an FLRW metric plus perturbations. These are among the features any averaging procedure should possess for it to be considered minimally adequate.

First, since the length scale represented by the local coordinate x' will always be much less than that represented by the cosmological coordinates x , we can assume that the volume of integration, and therefore the domain of x' , will be small with respect to the overall space-time coordinates x . Thus, we can expand the various functions of $(x+x')$ as multi-variable (because of the multiple coordinate variables, x' , y' , z' and t') Taylor series in these local coordinates, and truncate the series at first order. We can then write, for instance,

$$g_{\mu\nu FLRW}(x+x') \approx g_{\mu\nu FLRW}(x) + \frac{\partial g_{\mu\nu FLRW}}{\partial x^\lambda}(x) x'^\lambda. \tag{53}$$

Analyzing the first term in Eq. (52), we find that, in the Cartesian coordinate form of the metric specified in Eq. (42), $\langle g_{00}^I \rangle = 1$. Here and in the discussion which follows, we designate the four terms in Eq. (52) by the superscripts I, II, III, and IV, respectively. Further, using Eq. (53) we find also that

$$\langle g_{ij}^I \rangle = g_{ij FLRW} + \frac{\partial g_{ij FLRW}}{\partial x^\lambda}(x) \frac{\int d^4x' \sqrt{-g_{FLRW}(x+x')} x'^\lambda}{\int d^4x' \sqrt{-g_{FLRW}(x+x')}}. \tag{54}$$

These results for the first term of Eq. (52) enable us to determine what the application of our averager to an exact FLRW metric yields. If we carry out the integration indicated in the denominator over the averaging volume given in the local coordinates x'^λ , we find that this term vanishes, since the integral is odd in x'^λ . Thus, only even terms of the Taylor series will yield non-zero contributions to the average. Thus, to first order in the local coordinates, averaging an FLRW metric yields an FLRW metric. This result is trivial in the case of averaging a flat FLRW metric, but is not at first sight obvious for open and closed FLRW metrics. This result is somewhat important, because, as Boersma [6] has emphasized, we do not want the averaging procedure to introduce perturbations – we want the average of an FLRW metric to be an FLRW metric!

If we now go back to Eq. (52), we see that, since the terms II, III and IV will all be smaller than the pure FLRW term I, we have the further almost trivial result that

$$\langle g_{\mu\nu} \rangle = g_{\mu\nu FLRW} + \text{Perturbations}. \quad (55)$$

Though these are expected and obvious results, they are very reassuring. Our averaging procedure is physically and mathematically consistent. If it did not fulfil these conditions, it would have to be abandoned.

Studying terms II, III and IV of Eq. (52) in more detail, we find similarly that to first order

$$\langle g_{\mu\nu}^{II} \rangle = \frac{\int d^4 x' \sqrt{-g_{FLRW}(x+x')} h_{\mu\nu}(x+x')}{\int d^4 x' \sqrt{-g_{FLRW}(x+x')}} \approx h_{\mu\nu}(x) + \frac{\partial h_{\mu\nu}}{\partial x^\lambda}(x) \frac{\int d^4 x' \sqrt{-g_{FLRW}(x+x')} x'^\lambda}{\int d^4 x' \sqrt{-g_{FLRW}(x+x')} x'}, \quad (56)$$

and that

$$\langle g_{\mu\nu}^{III} \rangle = - \langle g_{\mu\nu}^{IV} \rangle \quad (57)$$

to first order in h and x' .

Then Eq. (52) becomes, to first order in the local coordinates,

$$\begin{aligned} \langle g_{\mu\nu}(x) \rangle &= g_{\mu\nu FLRW}(x) + \frac{\partial g_{\mu\nu FLRW}}{\partial x^\lambda}(x) \frac{\int d^4 x' \sqrt{-g_{FLRW}(x+x')} x'^\lambda}{\int d^4 x' \sqrt{-g_{FLRW}(x+x')}} + \\ &h_{\mu\nu}(x) + \frac{\partial h_{\mu\nu}}{\partial x^\lambda}(x) \frac{\int d^4 x' \sqrt{-g_{FLRW}(x+x')} x'^\lambda}{\int d^4 x' \sqrt{-g_{FLRW}(x+x')}}, \end{aligned} \quad (58)$$

However, as we just saw above, the integrals in the numerators of the second and fourth terms of Eq. (58) vanish when taken over the entire averaging domain, because they are odd in the local coordinates. Thus, we obtain the somewhat mysterious and apparently trivial result

$$\langle g_{\mu\nu}(x+x') \rangle = g_{\mu\nu FLRW}(x) + h_{\mu\nu}(x), \quad (59)$$

to first order in the local coordinates, just what we began with in Eq. (41). It appears that the averaging, to first order in the local coordinates has no effect – that the $h_{\mu\nu}$ in Eq. (59) is the same as that in Eq. (41). This requires some careful comment.

Eq. (59) is simply the consequence of terminating the Taylor series in the local coordinates with the linear term. Any further precision using this approximation resides in the higher-order terms. In particular, any further correction to the already small $h_{\mu\nu}(x)$ will be smaller than it already is (that is, of higher order than either $h_{\mu\nu}(x+x')$ itself or the value of the local coordinates x'^λ). In general, we should expect that $|\langle h_{\mu\nu}(x+x') \rangle| \leq |h_{\mu\nu}(x)|$, but the possibility of it being $< |h_{\mu\nu}(x)|$ can only be explored by either performing the averaging without approximation, or taking the Taylor series expansion of the metric to higher orders in x'^λ .

It does not make any physical sense to go to higher orders to determine the distortions averaging introduces in the FLRW background metric itself (in the closed and open cases – if we confine ourselves to averaging over spatial hypersurfaces, it is only in the closed and open cases that higher order distortions are introduced by averaging), since it does not vary significantly over local-coordinate length scales; it is already smooth on all scales up to those of the cosmological coordinates. It is intuitively clear that averaging over an FLRW metric should give an FLRW metric. From one point of view, we might almost say that the FLRW part of the metric should not be averaged – it has in a sense already been averaged, or rather is the result of an average having been already performed, or assumed – in particular, an average of the density used in the field equations.

With respect to the perturbed part of the metric $h_{\mu\nu}(x+x')$, however, there is a good reason to carry out the averaging to higher precision. In practice, it may, unlike the FLRW metric itself, vary a great deal on local length scales. Averaging over those local variations may significantly decrease its magnitude on cosmic length scales, even reducing it to zero. In fact we know that initially there were perturbations on all scales above that given by photon diffusion (Silk damping). And at our epoch we see that there are very large density fluctuations (much larger than perturbations!) on all local and intermediate scales – density enhancements as well as voids. In both cases the perturbations on cosmological scales are really the averages over the metric fluctuations on smaller scales.

Thus, when we write the perturbed FLRW metric as in Eq. (41) without further specification, we are in a sense conflating two very different cases, which must be treated separately. The first is that in which the metric of the universe on *all* scales can be represented as an FLRW metric plus small deviations, which are present on a large range of scales. In this case, the expression given in Eq. (41) is correct on all scales. But, at the same time it can be restricted to a given scale of interest – for instance, the cosmological scale. In doing that, we would naturally define the perturbed component $h_{\mu\nu}(x)$ as

$$h_{\mu\nu}(x) = \langle h_{\mu\nu}(x + x') \rangle, \quad (60)$$

where the $h_{\mu\nu}(x + x')$ are small on all length scales and where the averaging procedure is well-defined as applied to an FLRW metric plus perturbations: As long as the metric deviations from the FLRW metric on all scales are small, the averaging of $h_{\mu\nu}(x + x')$ over a volume of any size can be performed in a meaningful way, as we have indicated above. That is, the innumerable small metric fluctuations on local and intermediate scales can be averaged over to obtain those on cosmic scales. This would be the situation in the period of the universe up until density perturbations on one scale or another go nonlinear. It would certainly pertain to the epoch of recombination when the cosmic microwave background radiation (CMWBR) was last scattered, and to some hundreds of millions of years afterwards. Thus, it has important applications to CMWBR anisotropies.

The second situation would be as at present, when the metric of the universe can be represented by the perturbed FLRW metric in Eq. (41) only on cosmic length scales. On local and intermediate scales Eq. (41) is far from correct – the metric on these scales is nowhere near FLRW, deviations from FLRW being very large. Thus, the interpretation of Eq. (41) in this case is that on scales much larger than that of any nonlinear density fluctuation, the universe is close to FLRW, if the $h_{\mu\nu}(x)$ – the deviations from FLRW on *that* scale – are small. There will be lower limit on this length scale, below which this will not be true. Then this $g_{\mu\nu FLRW}(x) + h_{\mu\nu}(x)$ must be understood as the average of all the small-scale metrics – some of which will be very different from the cosmological “background metric” – over a volume of cosmological length scale. On small and intermediate length scales we will not be able to define a background, such that the actual metric is always a perturbation (or small fluctuation) with respect to it. How is this $g_{\mu\nu FLRW} + h_{\mu\nu}(x)$ to be calculated from the large amplitude small and intermediate scale fluctuations? And how should we implement our averaging procedure in this case? This is an important question and will be treated briefly in the next section.

In this second case, the averaging procedure we have outlined here is not at all adequate. But an adequate one must be constructed, if $h_{\mu\nu}(x)$ – the deviation of the large-scale metric from FLRW – is to have any meaning with respect to the metric fluctuations on smaller scales, many of which are not at all small. What is usually implicitly assumed in using $h_{\mu\nu}(x)$ in this context is to consider that we have averaged over the small and intermediate scale density inhomogeneities to obtain a large-scale density inhomogeneity which is found to be only marginally different from the background FLRW density and is therefore the source of a very small large scale $h_{\mu\nu}(x)$. However, this uncritically avoids a number of important mathematical and physical issues which really have to be resolved before such a procedure is validated – particularly the effective field equations which are really operative in averaging over large amplitude inhomogeneities, including the contribution made by gravitational binding energy to the average, the background metric which is used in the averaging process, and the adequacy and validity of any averaging process itself involving large deviations from either Minkowski or FLRW [1,5,7,12]. We need to apply our averaging procedure in these situations and see what it yields. In some simple, idealized cases an approximate averaging procedure such as that suggested by by Zotov and Stoeger [5] may, as we point out below, be equivalent to ours.

From this brief discussion, we begin to appreciate how different these two cases are, from the point of view of averaging – and the rather different problems implicit in each of them. As we have seen, it is only the first case, involving perturbations on all scales which can be dealt with using the linearized procedures developed so far, or any procedures limited to perturbations.

It is important to note, furthermore, that, as we have already seen, it is possible that in either of these two cases, the average of the perturbations or deviations over a large enough scale will yield zero, as in some of the weak-field-limit cases. In fact, strictly speaking, if a given FLRW background model is a genuine background, there should be a large enough scale over which averaging yields the exact FLRW model itself – that is, the average over the perturbations on such a large length scale gives zero. This means that the “positive” and “negative” perturbations need to balance on the largest scales, if the universe is really FLRW above a certain length scale. If this is *not* the case, then we must

choose the average FLRW plus perturbations as the real background. It is also clear, of course, that generally the average of the perturbations will possess much more symmetry than the original perturbations themselves.

C. Deviations in the Hubble Parameter

Of course, these issues of averaging will have consequences for the observational parameters we measure, for example the Hubble parameter H . It can be easily shown from a perturbation treatment of the field equations that, for a dust equation of state, the deviation of the Hubble parameter from its value in a best-fit FLRW model is just given by [22]

$$\Delta H = -\frac{1}{3}\dot{\delta}, \quad (61)$$

where δ is the density contrast, that is $\delta = \frac{\rho - \rho_b}{\rho_b}$, where ρ is the mass-energy density and ρ_b is the background (FLRW) mass-energy density.

Now, obviously Eq. (61) depends on how δ is calculated, and furthermore we want the time-derivative of the *cosmological* δ – its average over the length scale of the universe. Averages over small or intermediate scales will give us only the local or intermediate deviations of the Hubble parameter from its FLRW value. Similar to our discussion of averages over metric fluctuations in the two general cases in the last subsection, determination of δ and therefore $\dot{\delta}$ is relatively easy in the case when density fluctuations on all scales are perturbations, as long as we have the required data. Then we can use our procedure to do the averaging. But again, if we are faced with the second case, where the density inhomogeneities on small and intermediate scales are large, as at the present time, the correct averaging procedure of the density fluctuations is unclear. In fact, in this case, it is not certain that as a consequence of averaging the field equations at smaller scales over larger scales Eq. (61) holds. Not only that, but the perturbative treatment from which Eq. (61) is derived will not be valid in that regime. This issue will be investigated briefly in the next section, and more thoroughly in a subsequent paper. Furthermore, it is obvious again that the scale over which the averaging is done will determine the result. If we average over intermediate scales – or determine H using data sampling intermediate scales, scales at which the Hubble flow can not yet be recovered, instead of cosmological scales – then our calculation or measurement of H will be local, not cosmological.

VI. AVERAGING IN SIMPLE NONPERTURBATIVE CASES

When faced with averaging over large amplitude small and intermediate scale perturbations we can in principle still apply our averaging procedure. However, in general it is difficult to see how we could practically implement it directly. This is true even in simple idealized cases, where we have, for instance, strong spherically symmetric inhomogeneities. However, in such idealized situations, Zotov's and Stoeger's two-step procedure [5] provides a useful and implementable approximation to ours.

They first average over each individual spherically symmetric inhomogeneity, which is represented either as a Schwarzschild metric or a local FLRW metric plus a surrounding underdense annulus. This average is performed over the whole vacuole representing the inhomogeneity, without including any region outside it, and is therefore referred to the center point of the inhomogeneity.

Now, very large regions of the universe can be ideally considered to be made up of collections of these spherically symmetric inhomogeneities, or spherically symmetric clusters of them, separated by regions which are either approximately Minkowski – if the region is not expanding – or approximately FLRW, if the region is expanding. These function as a temporary or intermediate background metric, representing the metric far from the center of any inhomogeneity or cluster of inhomogeneities. Thus, the picture is very much like the Swiss-cheese model, except that we envision the background as temporary or auxilliary – to enable us to perform the second step in the averaging.

The second averaging is performed over these very large regions – usually of cosmological length scale – consisting of many spherically symmetric inhomogeneities. It is approximated by:

$$\bar{g}_{\mu\nu}(x) \cong \frac{1}{V_2} [\langle g_{\mu\nu} \rangle V_1 N + g_{\mu\nu B}(V_2 - V_1 N)], \quad (62)$$

where $\langle g_{\mu\nu} \rangle$ is the average over a single spherically symmetric homogeneity, V_1 is the volume of each homogeneity, and N is the number of them in the full cosmological averaging volume V_2 . $g_{\mu\nu B}$ is the auxiliary background metric. In this above equation we have idealized all the inhomogeneities as identical in volume and mass. We can easily generalize to many different sizes. x is the cosmological coordinate at which the second averaging is centered.

Thus, the second averaging can be conceived as done at each value of the cosmological coordinate x . We simply move the volume V_2 around the universe – to all different values of x and perform the two-step averaging procedure. It can be easily seen that this is equivalent to what our averaging procedure involves, precisely in terms of moving the same volume around the universe and performing an average over it. Thus it may be considered as a way of approximating what our procedure would give in these situations of strong localized inhomogeneities. As Zotov and Stoeger point out, $\bar{g}_{\mu\nu}(x)$ will generally not be an FLRW metric, even if it does not depend on the spatial coordinates – it will superficially look like FLRW, but the scale factor will have a different dependence on time than the FLRW metric. In general, of course, the result of the averaging will depend on the spatial cosmological coordinates.

Only if the averaging is over a large enough cosmological volume and the universe is such that averaging over that volume centered at each point in a spacelike slice of the universe gives exactly the same result, will it yield a scale factor independent of the spatial coordinates. If the dependence of the average scale factor on the spatial coordinates is weak and its time dependence not too much different from FLRW, then the average metric may be represented by a perturbed FLRW metric $g_{\mu\nu FLRW}(x) + h_{\mu\nu}$, where now $h_{\mu\nu}$ is a large length-scale perturbation from a FLRW metric which is defined via the above averaging procedure. However, strictly speaking, we should just treat $\bar{g}_{\mu\nu}(x)$ as the average metric over that length scale, since it will no longer exactly satisfy the Einstein field equations. [5] It will satisfy field equations which are Einstein's, with an extra term added, due to the noncommutability of averaging and forming the Einstein tensor from the metric.

Is such averaging approximately tensorial? It seems that it should be, with respect to the cosmological coordinates. For the prescription for averaging does not depend on the coordinate system used for the cosmological coordinates themselves. But do significant problems arise with regard to the implied averaging over local coordinates? And is the Zotov-Stoeger averaging over the moving volume really approximately equivalent to what our averager would give? We shall investigate these questions, along with others in a subsequent paper.

VII. CONCLUSIONS

In this paper we have proposed an intuitively clear averaging procedure for general relativity and cosmology, which is an extension to that used in electromagnetism. We have shown that it gives approximately unique and tensorial results in weak field and perturbed FLRW cases, and does not lead to any significant unacceptable results in these cases. Furthermore, it promises to be easily applicable in cases where fluctuations on all scales are perturbations, such as up to the epoch in which density perturbations begin to go nonlinear.

Finally, we explore the limits of averaging perturbed FLRW universes, indicating that averaging over very large small and intermediate scale inhomogeneities, in order to recover the average perturbation on large scales, requires applying our averaging procedure beyond the simple FLRW plus perturbations case. We show how that can be done approximately in simple idealized cases involving spherically symmetric objects and clusters of objects in an intermediate background using the Zotov and Stoeger approach.

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